

KCL-TH-94-19
hep-th/9411029
October, 94

Physical States and String Symmetries

Peter West
Mathematics Department
King's College
Strand
London WC2R 2LS

Abstract

It is shown that if the momenta belong to an integral lattice, then every physical state of string theory leads to a symmetry of the scattering amplitudes. We discuss the role of this symmetry when the momenta are those provided by the usual D.D.F construction and show that the string compactified on the torus associated with the self-dual Lorentzian lattice, $\Pi^{25,1}$ possess the Fake Monster Lie algebra as a symmetry.

Even from the earliest attempt [1] to construct an interacting gauge covariant string theory, it was clear that the string possesses a symmetry that mixed mass levels and that the string three vertex could be regarded as the structure constants of this symmetry. However, even though we now possess consistent gauge covariant string theories of the open [2] and closed [3] bosonic string, we lack a clear understanding of the symmetry principle that underlies string theory. One suggestion [20] was that the vertices associated with the algebra of physical states based on the unique self-dual Lorentzian Lattice in 26 dimensions could play a role.

In the group theoretic approach to string theory [5], the scattering amplitudes are uniquely determined by a set of simple equations, called overlap equations. These equations related how the operators in the theory of a given conformal weight act on the string scattering vertices. The overlap equations can be used to derive equations, sometimes called integrated overlap equations, that relate the contour integrals around Koba-Nielsen points of operators of conformal weight d with suitable $1 - d$ forms. For operators of conformal weight one, the required forms are functions of which a special case is the constant. It has been suggested [5] that such overlap relations may be regarded as symmetries of the string theory. Indeed, the physical state tachyon vertex operator was used [11] in this way to show that certain string vertices satisfied the Hirota equation. More recently [21], the conditions that the conformal weight one operators, corresponding to the physical vertex operators, imposed on the string vertex were considered when they were used in the above mentioned equation. It was shown [21] that, with some additional assumptions, they determined the tree level tachyonic scattering amplitude.

In this paper we further explore this approach to symmetry in string theory. We show, using the string three vertex, that for any physical state one can define a vertex operator which, when integrated, can be used to construct a symmetry of the string scattering amplitudes. For this result to hold, however, the action of the integrated vertex operator on the string scattering vertex must be well defined. In general this is not the case, but it holds if the momenta belong to an integral lattice. There are at least two interesting situations where this is true; one is when the string is compactified on a torus and the second is when we work with a string whose corresponding point particles have their momenta of the form $p - nk$ where $p \cdot p = 2$, $k \cdot k = 0$, $p \cdot k = 1$. This latter case arises in the well known D.D.F construction of the physical states.

The states of a quantum point particle belong to an irreducible representation of the Poincaré group. Indeed, one frequently turns this around and regards the quantum point particle as being defined as an irreducible representation of the Poincaré group. The usual

bosonic string contains an infinite number of irreducible representations of the Poincaré group. The string symmetry group can be viewed as the symmetry group which organizes all these representations into a single multiplet which comprises the string. Since the early days of string theory it has been known that the physical states of the bosonic string are generated by the spectrum generating algebra of the D.D.F operators [4]. The property of these operators that made them so useful was that one could act with one of them on any physical state and produce another physical state. In this paper, we show that the D.D.F operators acting on the physical states can also be interpreted as a symmetry of the scattering amplitudes. As such, they lead to symmetries of the string theory itself. Although the action of the D.D.F operators on the string scattering vertices is well defined in the usual Lorentz frame used to construct the corresponding states it is not well defined in any Lorentz frame. Hence the symmetry only holds for particles with momenta on the lattice associated with the D.D.F construction. Possible ways of extending the construction to arbitrary Lorentz frames are discussed.

Another situation in which one finds the above symmetry is valid is on string compactifications on a torus. A particularly interesting case is the completed compactification of the string on the torus corresponding to the unique self-dual Lorentzian lattice $\Pi^{25,1}$. In this case the string theory possess the Fake Monster algebra as a symmetry group and the N string scattering vertex and the N string S-matrix itself can be viewed as an invariant tensor and a kind of Casimir of this group respectively.

The calculations in this paper are carried out in the group theoretic approach [5]. The principal object in this approach is the scattering vertex. For tree level scattering, the scattering vertex $V^{(N)}$ is related to the scattering of N strings $|\chi\rangle_j$, $j = 1, \dots, N$ by

$$W^{(N)} = \int \prod'_i dz_i V^{(N)} \prod_{j=1}^N |\chi\rangle_j . \quad (1)$$

where z_i are the Koba-Nielsen coordinates. The N string vertex is defined to satisfy the “overlap equations”

$$V^{(N)} R^{(j)}(\xi^j)(d\xi^j)^d = V^{(N)} R^{(i)}(\xi^i)(d\xi^i)^d \quad (2)$$

for any operator of conformal weight d . These imply the integrated “overlap equations”

$$V^{(N)} \sum_{j=1}^N \oint d\xi^j R^{(j)}(\xi^j) \phi = 0 \quad (3)$$

where ϕ is a $1 - d$ form. Strictly, one should define the vertex using equation (2) only for the basic operators of the theory, however it has been found that one can then show

equation (2) then follows for all the conformal operators of the theory usually considered. The above equation also holds for loop scattering vertices, but in this case, the vertex also obeys additional overlap equations, one for each loop.

The physical states of the bosonic string satisfy

$$Q|\varphi\rangle = 0 \quad , \quad b_0|\varphi\rangle = 0 \quad (4)$$

The non-trivial cohomology classes belong to the states [6]

$$\psi c|0\rangle \quad , \quad |0\rangle \quad (5)$$

where $|0\rangle$ is the $SL(2, \mathbf{R})$ invariant vacuum and ψ is a functional of x^μ alone which is subject to $L_n^x \psi = 0 \quad n \geq 1 \quad ; \quad (L_0^x - 1)\psi = 0$.

Given any physical state $|\varphi\rangle$ we can use the three string vertex $V^{(3)}$, with ghost included [7], to form a corresponding operator; we act with $|\varphi\rangle$ on leg three to form

$$V_\varphi(z) \equiv V^{(3)} \Omega^{(2)} z^{L_0^{(1)}} z^{-L_0^{(2)}} (b_0^{(1)} - b_0^{(2)}) |\varphi\rangle_3 \quad (6)$$

where $\Omega^{(2)}$ is the twist operator applied on leg two, whose precise form [8] depends on the choice of local coordinates ξ^j used to define the vertex. For the old Canneschi, Schwimmer, Veneziano vertex [9], $\Omega = e^{-L_1} e^{i\pi L_0}$. Since the B.R.S.T. charge is the integral of a weight one conformal current, the N vertex satisfies

$$V^N \sum_{j=1}^N Q^{(j)} = 0. \quad (7)$$

Using the relation $\{b_n, \psi\} = L_n$, one finds that the above equation implies that

$$V_\varphi^{(2)}(z)(Q^{(1)} + Q^{(2)}) = z \frac{d}{dz} V_\varphi^{(2)}(z) \quad (8)$$

Clearly, the vertex $V_\varphi^{(2)}$ also satisfies $V_\varphi^{(2)}(z)(b_0^{(1)} - b_0^{(2)}) = 0$. From this two vertex we can define an operator $V_\varphi(z)$ by turning around line 1 and then identifying the 1 and 2 lines:-

$$\langle \phi | V_\varphi | \chi \rangle = V_\varphi^{(2)} | \phi \rangle_1 | \chi \rangle_2 \quad (9)$$

for all $|\phi\rangle$ and $|\chi\rangle$. The relations above then become

$$[V_\varphi(z), Q] = z \frac{d}{dz} V_\varphi(z) \quad (10)$$

$$[b_0, V_\varphi(z)] = 0 \quad (11)$$

Consequently, the operator

$$W_\varphi = \oint \frac{dz}{z} V_\varphi(z) \quad (12)$$

commutes with Q and b_0 . It follows that W_φ acting on a physical state takes it to another physical state. If the physical state φ is of the form $|\varphi\rangle = Q|\Lambda\rangle$ then

$$W_\varphi = [Q, W_\Lambda] \quad (13)$$

A consequence of the fact that the physical state $|\varphi\rangle$ is annihilated by L_n ; $n \geq 0$ is that $V_\varphi(z)$ is an operator of conformal weight one. To demonstrate this result one considers the integrated overlap of equation (3) taking $R = T$ and ϕ to be a vector field that is analytic everywhere except at the Koba-Nielsen points associated with legs one and two, where it may have poles, and it also vanishes at the Koba-Nielsen point corresponding to string three.

One could derive analogous results for an operator V'_φ , obtained by the above procedure with the exception that one did not insert in equation (6) the $(b_0^{(1)} - b_0^{(2)})$ factor. This operator, V'_φ will commute with Q and be of weight zero. These results are consistent with the association between physical states and the vertex operator used in the conformal field theory approach to scattering in string theory.

The above analysis used a formalism which included ghosts in all the vertices and Virasoro generators. We can also work without ghosts, that is in the old covariant formalism. Following almost identical arguments one finds a V_φ that is an operator of conformal weight one. Its integral W_φ takes physical states to physical states. For the rest of this paper we will, for convenience, work in the formalism without ghosts since in this case the scattering amplitudes, as shown in equation (1), do not require ghost insertions which complicate the form of the overlaps.

It now follows from the overlap of equation (3) that for any physical state $|\varphi\rangle$, the N string vertex obeys the equation

$$V^{(N)} \left\{ \sum_{j=1}^N W_\varphi^{(j)} \right\} = 0 \quad (14)$$

What is not immediately apparent however, is that this equation only holds if the contour integrals contained in W_φ are well defined. This will be the case if the action of $V_\varphi(z)$ on $V^{(N)}$ induces only poles in $z - z_j$ instead of more singular behavior.

Examining the form of N string scattering of equation (1) one finds that it is invariant under

$$\delta|\chi\rangle = W_\varphi|\chi\rangle \quad (15)$$

provided the action of W_φ on $|\chi\rangle$ is also well defined. Consequently, it is a symmetry of both the states and the scattering amplitudes and so is a symmetry of string theory.

The most usual variation of a field found in quantum field theory is the product of the field itself, a representation of the generator of the group and the parameter. The analogue with the variation found above is to interpret the physical state as the parameter and the three vertex as the group representation of the generator. Since the physical states lead to the generators of the group and we act with these on physical states, we are using an adjoint action of the group and so we should regard the three vertex as the structure constants.

It is instructive to derive equation (4) for the photon states. It is straight-forward to follow the procedure leading to equation (12) taking φ to be the photon state. One finds that the corresponding W_φ 's are the D.D.F operators [4],

$$A_n^i(k) = \oint dz e^{ink \cdot Q(z)} \mathbf{P}^i(z). \quad (16)$$

where $\mathbf{P}^i = \partial Q^i$, $k^2 = 0$ and we have chosen to work in the Lorentz frame where $k^i = 0$. The N string vertex obeys the overlap equation

$$V^{(N)} Q^{(i)\mu}(\xi^i) = V^{(N)} Q^{(j)\mu}(\xi^j) \quad (17)$$

Differentiating with respect to ξ^i we find that

$$V^{(N)} \mathbf{P}^{(i)\mu}(\xi^i) = V^{(N)} \mathbf{P}^{(j)\mu}(\xi^j) \frac{d\xi^j}{d\xi^i}. \quad (18)$$

Equation (17) implies that

$$\begin{aligned} & V^{(N)} Q^{(i)\mu_1}(\xi_1) Q^{(i)\mu_2}(\xi_2) \dots Q^{(i)\mu_n}(\xi_n) \\ &= V^{(N)} Q^{(j)\mu_n}(\xi_n) \dots Q^{(j)\mu_1}(\xi_1). \end{aligned} \quad (19)$$

If we consider $k_\mu Q^\mu$ for $k^2 = 0$, we have no normal ordering problems and

$$V^{(N)} [ik \cdot Q^{(i)}(\xi^i)]^m = V^{(N)} [ik \cdot Q^{(j)}(\xi^j)]^m \quad (20)$$

Dividing by $m!$ and summing we find that

$$V^{(N)} \exp[ik \cdot Q^{(i)}(\xi^i)] = V^{(N)} \exp[ik \cdot Q^{(j)}(\xi^j)] \quad (21)$$

Applying equation (18) and taking $k \rightarrow nk$ we find that the photon emission operator obeys the equation

$$V^{(N)} e^{ink \cdot Q^{(i)}(\xi^i)} \mathbf{P}^{(i)k}(\xi^i) = V^{(N)} e^{ink \cdot Q^j(\xi^j)} \mathbf{P}^{(j)k}(\xi^j) \frac{d\xi^j}{d\xi^i} \quad (22)$$

Integrating this overlap equation we find that

$$V^{(N)} \sum_{j=1}^N A_n^{(j)i} = 0. \quad (23)$$

Since the D.D.F operators are the contour integral of the physical photon emission operator, which is an operator of conformal weight one, they commute with the L_n . Consequently, the transformation $\delta|\chi\rangle = A_n^k|\chi\rangle$ takes us from one physical state to another, and equation (23) tells us that this is also a symmetry of the string scattering amplitudes.

One can also establish [11] a related result to equation (21) for any momentum k^μ , whereupon one finds an additional factor of $(\frac{d\xi^j}{d\xi^i})^{\frac{k^2}{2}}$. Consequently, for $k^2 = 2$, ie tachyons, one finds an equation similar to equation (23), except with no \mathbf{P}^k , and so an analogous symmetry.

It is well known, at least for $D=26$, that all the physical states with positive norm are of the form

$$\prod_{\{\lambda_{n,i}\}} (A_{-n,i}^i)^{\lambda_{n,i}} |0, p'\rangle \quad (24)$$

where $(p')^2 = 2$, $p' \cdot k = 1$. Since, the D.D.F states generate all states it follows that equation (23) holds for all physical states. Although one can work with D.D.F operators alone, it is perhaps more natural to work with all the physical states as we first did.

To clarify the nature of the above symmetry, it is instructive to recall the analogous steps for the point particle. The quantum point particle transforms as an irreducible representation of the Poincaré group. The unitary irreducible representations of the Poincaré group are found using the method of induced representations which we now summarize for the massless case. Given a particle of momentum k^μ , $k^2 = 0$ we can choose its components to be $k^- = \frac{1}{\sqrt{2}}(k^{D-1} - k^0) = 1$, $k^+ = k^i = 0$. This choice is preserved by the “little group” $ISO(D-2)$ generated by the momentum operators P^μ , J^{ij} ; $i, j = 1, \dots, D-2$ and J^{i+} . We now choose an irreducible unitary representation of this group. Since the J^{i+} generate a non-compact group, any unitary representation, φ must be trivial under

these generators; $J^{+i}\varphi = 0$. One is then left with $SO(D-2)$. The representation of the full Poincaré group is then found by boosting the φ according to the usual induced procedure. We can, for example, choose the fundamental representation, φ^i , $i = 1, \dots, D-2$ which corresponds to the photon.

Physicists usually find it useful not to work with the above representation, but to extend it by embedding it into a larger representation. For the photon, we consider φ^μ ; $\mu = 0, 1, \dots, D-1$ which transforms in the fundamental representation of $SO(D-1, 1)$. For massless particles, this embedding can be performed in two steps. In the first step, we consider the $SO(D-2)$ invariant subspace given by $\varphi^{\hat{\mu}} = \{\varphi^i, \varphi^-\}$. In this space we release the unitarity restriction mentioned above, but replace it by the equivalence relation $\varphi^{\hat{\mu}} \sim \varphi^{\hat{\mu}'}$ if $\varphi^{\hat{\mu}} = \varphi^{\hat{\mu}'} + k^{\hat{\mu}}\Lambda$. This is just the gauge transformation which in effect removes φ^- . Finally, we can consider the full space φ^μ , but impose the condition $k \cdot \varphi = 0$ which restricts one to the formulation just discussed.

Let us now consider the analogous steps for string theory. It is well known [10] that the D.D.F operators admit a Lorentz extension

$$A_n^\mu(k) = \oint : e^{ink \cdot Q(z)} \mathbf{P}^\mu(z) : dz - \frac{n}{2} k^\mu F_n(k) \quad (25)$$

where

$$F_n(k) = \oint dz \frac{k \cdot : \partial \mathbf{P} :}{k \cdot \mathbf{P}} e^{ik \cdot Q} : \quad (26)$$

These operators obey the algebra

$$\begin{aligned} [A_n^\mu(k), A_m^\nu(k)] = & n \delta_{n+m,0} P \cdot k \eta^{\mu\nu} + m k^\mu A_{n+m}^\nu(k) \\ & - n k^\nu A_{n+m}^\mu(k) + n^3 k^\mu k^\nu \delta_{n+m,0} P \cdot k \end{aligned} \quad (27)$$

where we have used the relation $k \cdot A_n(k) = \delta_{n,0} P \cdot k$.

One choice of little group is that generated by $P^\mu, A_n^i(k)$ and J^{ij} , $i, j = 1, \dots, D-2$ with $k^- = 1, k^i = k^+ = 0$. Acting on the momentum state $|p^\mu\rangle$ with momentum $p^+ = p^- = 1, p^i = 0$ with the A_n^i we generate all the physical states of the string with positive definite norm. We note that we can not include more of the Lorentz group since these will not leave both the massless vector k^μ and the massive vector p^μ inert.

We could also extend the choice of little group to be P^μ, A_n^i, A_n^- and J^{ij} . The generators

A_n^- obey the relations implied by equation (27), but it is preferable to work with

$$R_n = -A_n^- - \frac{1}{2} \sum_{i=1}^{D-2} \sum_{p=-a}^{\infty} : A_{n-p}^i A_n^i : - \frac{(D-2)}{24} \delta_{n,0} \quad (28)$$

since it obeys the simpler algebra

$$\begin{aligned} [R_n, A_m^i] &= 0 \\ [R_n, R_m] &= (n-m)R_{n+m} + \frac{n^3}{12}(26-D)\delta_{n+m,0} \end{aligned} \quad (29)$$

The R_n obey a Virasoro algebra with central charge $26-D$ which vanishes in $D=26$, a choice we now adopt. In a unitary representation, R_n must therefore be trivially realized and so one is left with the previous D.D.F. states. As for massless representations of the Poincaré group, we can extend the representation to include states which have non-trivial R_{-n} factors and consider two states as equivalent if they differ by a state that includes any R_{-n} 's acting.

To carry out the analogue of the final step, that is to obtain a manifestly Lorentz covariant formulation, we add to our operators, an operator $\phi_n(k)$, $k^2=0$ which satisfies

$$\begin{aligned} [\phi_n(k), A_m^\mu(k)] &= nk^\mu \phi_{n+m} \\ [\phi_n(k), \phi_m(k)] &= 0 \end{aligned} \quad (30)$$

A representation of ϕ_n is given by [10]

$$\phi_n = \oint \frac{dz}{z} e^{ink \cdot Q(z)}, \quad (31)$$

however, we can regard ϕ_n as an abstract operator at this point.

We now consider states in an enlarged configuration space of the form

$$|\{K, \lambda\}\rangle = \prod_{\{K_m\}} (\phi_{-m}(k))^{K_m} \prod_{\{\lambda_p\}} (R_{-p}(k))^{\lambda_p} \prod_{\{\lambda_{n,i}\}} (A_{-n}^i(k))^{\lambda_{n,i}} |0, p\rangle. \quad (32)$$

where

$$\phi_n |0, p\rangle = 0 = R_n |0, p\rangle = A_n^i |0, p\rangle, n \geq 1 \quad (33)$$

and $k \cdot p = 1$. Clearly, this enlarged representation contains the representation of equation (24) by simply omitting the ϕ_{-m} and R_{-p} factors.

In fact, all the states of equation (33) are linearly independent. The simplest way to demonstrate this fact is to note that the determinant of the matrix

$$M(\{K, \lambda\}, \{K', \lambda'\}) = \langle K, \lambda | K', \lambda' \rangle \quad (34)$$

is non-zero. The proof is similar to that given by Thorn [13], albeit for a different set of operators. Essentially, one can adopt an ordering of the oscillator factors such that M has non-zero entries on the diagonal from the lower left to the upper right and vanishing entries below this diagonal. This occurrence stems from the fact that

$$\langle 0, p | \phi_{n_1} \phi_{n_2} \phi_{n_3} \dots \phi_{n_s} | 0, p \rangle \quad (35)$$

vanishes unless $n_1 = n_2 = n_3 = \dots = n_s = 0$.

We can also consider the linearly independent states of the form

$$\prod_{\{\tau\}} (\alpha_{-n}^\mu)^{\tau_{n,\mu}} | 0, p \rangle \quad (36)$$

where $\alpha_n^\mu | 0, p \rangle = 0, n \geq 1$. The number of states at each level is determined only by the number of oscillator operating on the vacuum. There are therefore the same number of states in the Hilbert space whose states are the form given in equation (36) as in the Hilbert space whose states are of the form of those of equation (32). Indeed, using the explicit formulas for A_n^μ and ϕ_n in terms of α_n^μ we may construct the latter Hilbert space from the former.

To recover the original representation in this larger Hilbert space we must find the projection conditions. The long history of string theory has provided us with these objects. To project onto the states of equation (24) we can use the Brink-Olive projection operator [14], while if we wish to restrict only to the states of equation (32) with no ϕ_n 's we use the Virasoro conditions

$$(L_n - \delta_{n,0}) | \{K, \lambda\} \rangle = 0 \quad n \geq 0 \quad (37)$$

where $L_n = \frac{1}{2} \sum : \alpha_{n-p}^\mu \alpha_p^\mu :$

To demonstrate this we define the operators

$$\phi_{m,n} = \oint \frac{dz}{z} z^m e^{ink \cdot Q} \quad (38)$$

which have the following commutator with L_n ;

$$[L_n, \phi_{m,p}] = -(n+m) \phi_{n+m,p} \quad (39)$$

Acting with

$$(L_1)^{K_1} \dots (L_m)^{K_m} \quad (40)$$

on $|\{K, \lambda\} \rangle$ we can reduce all the ϕ_n 's to $\phi_{r,n}$'s which either annihilate due to the relation

$$\phi_{m,n}|0, p \rangle = 0 \quad (41)$$

unless $m+n \leq 1$ or, if $r+n=0$, give a factor one as a result of the relation $\phi_{-n,n}|0, p \rangle = |0, p \rangle$. Consequently, one is left with only the A_{-n}^μ oscillators acting on $|0, p \rangle$. However, according to equation (37) this should vanish which can only happen if there were no ϕ_{-n} factors in the original state. Hence, the Virasoro conditions do achieve the relevant projection conditions.

Thus we see there is a close parallel between the theory of induced representations of the Poincaré group for the point particle and the structure of the states on string theory. In the string theory case we have extended the little group algebra to include the ϕ_n 's. This allows us to construct the L_n 's which commute with the original algebra. The L_n constraints are usually derived from the Nambu action after first quantization, however, the above derivation provides a more algebraic derivation which it would be desirable to make more systematic.

The action of V_φ on the N string vertex is not in general well defined. If the string carries a momentum p_j on leg j and the physical state carries momentum k_j , V_φ will generate a term of the form $(\zeta^j - z^j)^{p_j \cdot k_j}$ when it acts on the scattering vertex. This term when integrated will only be well defined if $p_j \cdot k_j \in \mathbf{Z}$. If, however, all the states have momenta of the form $p - nk$, $n \in \mathbf{Z}$ then the action is well defined and we have a good symmetry. In the Lorentz frame in which the D.D.F operators are usually constructed, the momenta of the states do have the above form and so the above symmetry generated by physical states is a symmetry of the string theory in this Lorentz frame. In a general Lorentz frame the action of the symmetry generators is not well-defined since $p_j \cdot k_j$ is not an integer. It is perhaps not unreasonable to try to build a string theory in which this symmetry generalizes to all Lorentz frames. There are several approaches to this problem. One could drop the requirement of possessing commutation relations. An example of an algebra whose currents do have operator product expansions that involve only poles is that generated by parafermions [12], although even in this case the singularities in the operator product expansions are of the form $(z-w)^{p+\frac{q}{N}}$ where $p, q, N \in \mathbf{Z}$. An alternative is to try to extend the theory so as to still retain a Lie algebra, one such possibility is to add twisted fields [16] to the theory. It has been found [17] that one can construct projective invariant off-shell vertex operators, in 26 dimensions, using these fields and one may hope that these vertex operators have operator product expansions that can be used

to define Lie algebras. It is worth recalling that these fields were originally introduced in order to construct off-shell string scattering. One could also attempt to recover the known gauge covariant string theories from this starting point.

The symmetry will also be valid when one compactifies left and right sectors independently on a torus, since the associated momenta belong to $\Lambda \cap \Lambda^*$ where Λ and Λ^* are the associated lattice and its dual. A particularly, interesting case is the complete compactification of the closed bosonic string on the self-dual Lorentzian lattice $\Pi^{25,1}$. The algebra of physical states associated with this lattice has been studied in reference [15] and is referred to as the Fake Monster Lie algebra. This algebra is generated not only by a subset of the states corresponding to roots of length two, but also by those corresponding to integer multiples of the null vector $w = (0, 1, 2 \dots, 24; 70)$. One could construct the physical states using a D.D.F construction as was done for low levels for the algebra E_{10} [18]. Since all the momenta belong to $\Pi^{25,1}$ the action of the integrated vertex operator of equation (12) on the N string vertex is always well defined and as a result every physical states generates a vertex operator that is a symmetry of the scattering amplitudes. Hence, we conclude that the bosonic string compactified on the torus associated with the self-dual Lorentzian lattice $\Pi^{25,1}$ has the Fake Monster algebra as a symmetry. Indeed an N string vertex can be viewed as an invariant tensor and the N string S-matrix as a kind of Casimir of this algebra. Given that the vertices are uniquely determined by the Q^μ overlap equations it is perhaps not unreasonable to suggest that one could determine the properties of this string solely from demanding that the Fake Monster algebra be a symmetry. Some progress in this direction has been made in reference [21] and it would be interesting to investigate this suggestion further and perhaps use it to try to determine any non-perturbative behavior the string possess.

One might regard the symmetry discussed in this paper as part of the deeper symmetry that string theory is thought to possess. A useful analogy can be made with general relativity where one might suppose that prior to the discovery of general relativity physicists had a method of finding solutions to Einstein's equations without knowing what these equations were or understanding general coordinate invariance. One would find that certain special solutions possessed more symmetry than others, from our modern perspective these would result from the presence of Killing vectors, but from the perspective of the early physicists these additional symmetries might provide clues to the principle of general coordinate invariance [19]. The same might be true of the of the string compactified on the $\Pi^{25,1}$ lattice in that it could be used as a toy model to gain insights into string theory.

It would also be of interest to consider the $\alpha \rightarrow 0$ limit of the above algebra. Taking

this limit in the most naive manner implies that the D.D.F operators commute and as a consequence all the physical states except for the tachyon become null. While this calculation is very heuristic it does lead to the conclusion that the string at high energy has fewer degrees of freedom, in agreement with a number of other considerations.

Acknowledgement

I wish to thank Reinhold Gebert, Paul Howe and Hermann Nicolai for discussions.

References

- [1] A. Neveu and P. West Nucl Phys B268 (1986) 125.
- [2] A. Neveu and P. West Nucl Phys B293 (1986) 266,
E. Witten Nucl Phys B268 (1986) 253.
- [3] A. Neveu and P. West Nucl Phys B293 (1986) 266,
B. Zwiebach "Closed string field theory: an introduction", preprint hep-th/9305026.
- [4] E. Del Giudice, P. Di Vecchia and S. Fubini; Ann. Phys. 70 (1972) 378.
- [5] For a review of the group theoretic approach see P. West, Nucl. Phys. B (Proc. Suppl) 5B (1988) 217, Nucl. Phys. B320 (1989) 103.
- [6] M. Kato and K. Ogawa; Nucl. Phys. B212 (1983) 443,
A. Neveu, H. Nicolai and P. West; Phys. Lett. 175B (1986) 307,
M. D. Freeman and D. Olive; Phys. Lett. 175B (1986) 151.
- [7] A. Neveu and P. West; Nucl Phys B278 (1986) 601.
- [8] A. Neveu and P. West; Commun. Math. 119 (1988) 585.
- [9] L. Caneschi, A. Schwimmer and A. Veneziano, Phys. Lett, 30B (1969) 351.
- [10] R. C. Brower; Phys. Rev. D6 (1972) 1655.
- [11] B. Nilsson and P. West, Commun. Math. Phys. 145 (1992) 329.
- [12] A. Zamolodchikov Theor. Math. Phys. 65 (1989) 1205.
- [13] C. B. Thorn; Nucl. Phys. B248 (1984) 551.
- [14] L. Brink and D. Olive; Nucl. Phys. B56 (1973) 253.
- [15] R. Borcherds Advances in Mathematics 83 (1990) 30.
For a review see R. Gebert, Int. Jour. Mod. Phys. A8 (1993) 5441.
- [16] E. Corrigan and D. Fairlie, Nucl. Phys. B91 (1975) 397.
- [17] M. Bershadski; Int. Jour. Mod. Phys. A1 (1986) 443.
- [18] R. Gebert and H. Nicolai "On E_{10} and the D.D.F construction" preprint Desy 94-106.
- [19] This analogy was put to me by Hermann Nicolai.

- [20] A. Neveu and P. West; Phys. Lett. 179B (1986) 235.
- [21] G. Moore, "Symmetries of the Bosonic String S-matrix" hep-th/9310026 and hep-th/9404025.